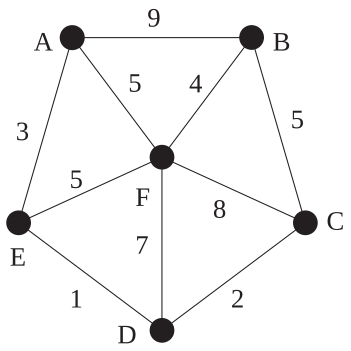
EXACT SOLUTION METHODS

* An instance: $I = (F, c)$.
* Suppose that:
  $f \in F = \mathbf{f} = [f_1, ..., f_n]^T$.
* Explicit/implicit constraints.
* Examples: TSP



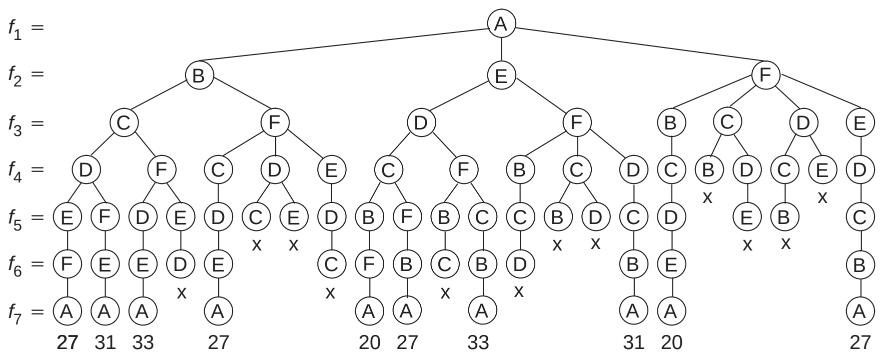
EXHAUSTIVE SEARCH: BACKTRACKING

```
backtrack(int k)
{
   float new_cost;
   if (k = n) {
      new_cost := cost(val);
      if (new_cost < best_cost) {
         best_cost := new_cost;
         best_solution := copy(val);
      }
   }
   else
      for each (el ∈ allowed(val, k)) {
         val[k] := el;
         backtrack(k + 1);
      }
}
```

```
float best_cost;
solution_element val[n], best_solution[n];

main ()
{
   best_cost := ∞;
   backtrack(0);
   report(best_solution);
}
```


SEARCH TREE EXAMPLE



BRANCH-AND-BOUND SEARCH

* Partial solution: $\tilde{\mathbf{f}}^{(k)}$.
* Cost estimation of a partial solution consists of cost components for:
  + specified part of solution,
  + unspecified part of solution.

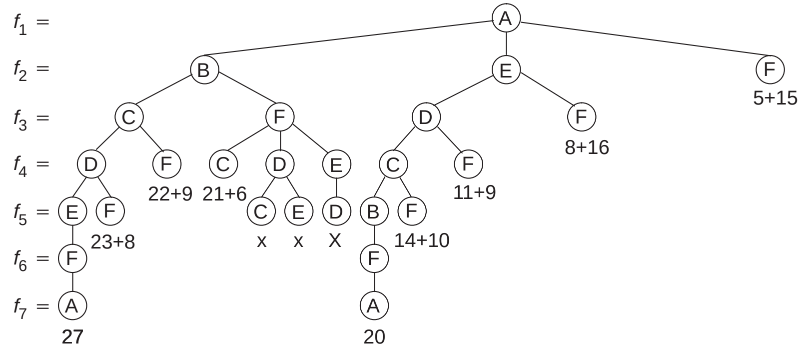
  $$\tilde{c}(\tilde{\mathbf{f}}^{(k)}) = \tilde{g}(\tilde{\mathbf{f}}^{(k)}) + \tilde{h}(\tilde{\mathbf{f}}^{(k)})$$

* In case of TSP: use *spanning tree* for estimation. The spanning tree is a minimal-weight tree in a graph. Consider in this case the all the points still to be interconnected; they total interconnection length will never exceed the length of the minimal spanning tree. The spanning tree can be found with a polynomial-time algorithm.

BRANCH-AND-BOUND SEARCH (CODE)

```
b_and_b(int k)
{
  float new_cost;
  if (k = n) {
    new_cost := cost(val);
    if (new_cost < best_cost) {
      best_cost := new_cost; best_solution := copy(val);
    }
  }
  else if (lower_bound_cost(val,k) ≥ best_cost)
        /* No action, node is killed. */
  else
    for each (el ∈ allowed(val, k)) {
      val[k] := el;
      b_and_b(k + 1);
    }
}
```


BRANCH-AND-BOUND EXAMPLE



BACKTRACKING VARIATIONS

* Use *breadth-first* search instead of *depth-first* search. Possibilities for *queue*:
  + FIFO,
  + LIFO,
  + Least cost.

* Use *dynamic* search tree instead of *static* search tree.

DYNAMIC PROGRAMMING

* Consider optimization problems characterized with a *complexity parameter* p (in general: multiple complexity parameters).
* Main idea: construct the optimal solution for some instance with $p = k$ using known solutions of instances with $p < k$; this is done by means of some *construction rule.*
* Use the construction rule to start building intermediate solutions starting from the smallest instances required (e.g. $p = 0$ or $p = 1$ and usually trivial to solve).

DYNAMIC PROGRAMMING FOR TSP

* Given is the graph $G(V, E)$ with edge weights w.
* Select an arbitrary vertex $v_s \in V$.
* $p = k$ means find shortest path from v_s to any $v \in V$ that goes through exactly k intermediate vertices.
* Notation: $C(S, v)$ is shortest path length from v_s to v exactly passing through the vertices in S.
* Solution amounts to computing:

$$C(V \backslash \{v_s\}, v_s).$$

* Construction rule:

$$C(S, v) = \min_{m \in S} \left[C(S \backslash \{m\}, m) + w((m, v)) \right]$$

INTEGER LINEAR PROGRAMMING (ILP)

* Special case of *linear programming*.
* General method to convert a large class of combinatorial optimization problems into a uniform mathematical form.
* After conversion, the problem can be solved by ILP-solvers.
* ILP is NP-complete.
* Applications in combinatorial optimization:
  + for small problem instances
  + to have certainty about exact solution for benchmarking heuristics
  + as a source of inspiration for developing new heuristics.

LINEAR PROGRAMMING EXAMPLE

* A company produces two products P_1 and P_2 with ingredients I_1 and I_2.
* P_1 uses a_{11} units of I_1 and a_{21} units of I_2. Its unit price is c_1. Its daily production is x_1 units.
* P_2 uses a_{12} units of I_1 and a_{22} units of I_2. Its unit price is c_2. Its daily production is x_2 units.
* The company cannot receive more than b_1 units of I_1 and b_2 units of I_2 per day.
* Problem: maximize the daily revenue $c_1 x_1 + c_2 x_2$ subject to

$$a_{11}x_1 + a_{12}x_2 \leq b_1 \qquad x_1 \geq 0$$
$$a_{21}x_1 + a_{22}x_2 \leq b_2 \qquad x_2 \geq 0$$

LINEAR PROGRAMMING (LP)

Given : matrix $\mathbf{A}$ vectors $\mathbf{b}$, $\mathbf{c}$ (constants) and the vector $\mathbf{x}$ (unknowns).

Canonical form:

* Minimize or maximize:

$$\mathbf{c}^T\mathbf{x}$$

* Subject to:

$$\mathbf{Ax} \leq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

Standard form:

* Minimize or maximize:

$$\mathbf{c}^T\mathbf{x}$$

* Subject to:

$$\mathbf{Ax} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

* The two forms can be converted into each other.
* Solvable in polynomial time by *ellipsoid* algorithm; in practice better performance with *simplex* algorithm (exponential time complexity).

INTEGER LINEAR PROGRAMMING

* Additional constraint on linear programming: all variables are integers.
* Solving the LP version first and then rounding results may give bad or unfeasible solutions.
* A special case that is often encountered is *zero-one* ILP: all variables can be either 0 or 1.

ILP FOR TSP

* Given is the graph $G(V, E)$ with edge weights w.
* Introduce a variable x_i for each edge $e_i \in E$, $1 \le i \le k$.
* $x_i = 1$ if and only if e_i is part of the solution.
* Cost function to minimize:

$$\sum_{i=1}^{k} w(e_i)x_i$$

* Constraints to ensure:
  + that only two edges per vertex are selected;
  + that there are no multiple disjoint tours.