THE CORDIC ALGORITHM AND CORDIC ARCHITECTURES

Implementation of Digital Signal Processing

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OUTLINE

- CORDIC algorithm:
  - Rotation and vectoring modes
- CORDIC architectures
- *Introduction to Project GFS*
- Applications of CORDIC

REFERENCES

- Andraka, R., "A Survey of CORDIC Algorithms for FPGA-Based Computers", *6th International Symposium on Field Programmable Gate Arrays*, Monterey, CA., pp. 191-200, (1998).
- Loehning, M., T. Hentschel and G. Fettweis, "Digital Down Conversion in Software Radio Terminals", *10th European Signal Processing Conference, EUSIPCO 2000*, pp. 1517-1520, (2000).

WHAT IS CORDIC?

- CORDIC: abbreviation of *coordinate rotation digital computer*.
- First publication by Volder, 1959.
- A method from the field of *computer arithmetic* allowing for the efficient implementation of a wide range of computations.

VECTOR ROTATIONS (1)

- Consider a sequence of rotations of a vector $(x^{(i)}, y^{(i)})^{\mathrm{T}}$ rotated by α_i to give vector $(x^{(i+1)}, y^{(i+1)})^{\mathrm{T}}$.
- So: $\begin{bmatrix} x^{(i+1)} \\ y^{(i+1)} \end{bmatrix} = \begin{bmatrix} \cos(\alpha_i) & -\sin(\alpha_i) \\ \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix} \begin{bmatrix} x^{(i)} \\ y^{(i)} \end{bmatrix}$
- After rewrite: $\begin{bmatrix} x^{(i+1)} \\ y^{(i+1)} \end{bmatrix} = \cos(\alpha_i) \begin{bmatrix} 1 & -\tan(\alpha_i) \\ \tan(\alpha_i) & 1 \end{bmatrix} \begin{bmatrix} x^{(i)} \\ y^{(i)} \end{bmatrix}$
- If $\tan(\alpha_i)$ is chosen such that $\tan(\alpha_i) = d_i 2^{-i}$, with $d_i = \pm 1$, then the rotations can be executed without multiplications except for initial factor $\cos(\alpha_i) = \frac{1}{\sqrt{1+2^{-2i}}}$

VECTOR ROTATIONS (2)

- If $\tan(\alpha_i) = d_i 2^{-i}$, this means: $\alpha_i = d_i \arctan(2^{-i})$
- For an arbitrary angle $\alpha, -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, the angle can then be decomposed as:

$$\alpha = \sum_{i=0}^{n} d_i \arctan(2^{-i})$$

- Angles involved:

i	0	1	2	3	4	5	6	7	8
2^{-i}	1	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
$\arctan(2^{-i})$[deg]	45.0	26.6	14.0	7.1	3.6	1.8	0.9	0.4	0.2

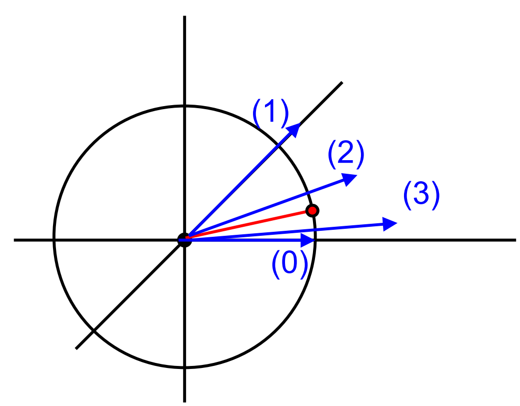
VECTOR ROTATION EXAMPLE (1)

- The 8 subsequent rotations for a rotation of 15 degrees are:

i	0	1	2	3	4	5	6	7	8
2^{-i}	1	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
$\arctan(2^{-i})$	45.0	26.6	14.0	7.1	3.6	1.8	0.9	0.4	0.2
d_i	1	-1	-1	1	1	-1	1	1	1
$\Sigma\alpha_i$	45.0	18.4	4.4	11.5	15.1	13.3	14.2	14.7	14.9

- The arctangent values can be precomputed and stored in a look-up table (LUT), say $L(i)$.
- The d_i depend on the required rotation angle.

VECTOR ROTATION EXAMPLE (2)



Note that the vector length is growing at each step.

CORDIC CONVERGENCE

- *Can one approximate an angle up to any degree of precision by increasing the number of rotation steps or is there any limit to the precision that can be achieved?*
- One can indeed achieve any degree of precision as the sum of the angles in steps $i+1$ to ∞ is larger or equal to the angle in step i for any i.
- Clear from table for small i.
- For large i, $\arctan(2^{-i}) \approx 2^{-i}$,

$$\sum_{n=i+1}^{\infty} 2^{-n} = 2^{-(i+1)} \sum_{n=0}^{\infty} 2^{-n} = 2^{-i}$$

ANGLE ACCUMULATION

- Keep track of total rotation angle in an *angle accumulator*:

$$z^{(i+1)} = z^{(i)} - d_i L(i)$$

- The angle accumulator can be used to determine d_i:
  - Initialize $z^{(0)} = \alpha$.
  - Factor d_{i+1} becomes 1 when $z^{(i)} \geq 0$ and -1 otherwise.

i	0	1	2	3	4	5	6	7	8	9
2^{-i}	1	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/252
$\arctan(2^{-i})$	45.0	26.6	14.0	7.1	3.6	1.8	0.9	0.4	0.2	0.1
d_i	-	1	-1	-1	1	1	-1	1	1	1
$z^{(i)}$	15.0	-30.0	-3.4	10.6	3.5	-0.1	1.7	0.8	0.3	0.1

CORDIC EQUATIONS SUMMARY

- Original equations were:

$$\begin{bmatrix} x^{(i+1)} \\ y^{(i+1)} \end{bmatrix} = \cos(\alpha_i) \begin{bmatrix} 1 & -\tan(\alpha_i) \\ \tan(\alpha_i) & 1 \end{bmatrix} \begin{bmatrix} x^{(i)} \\ y^{(i)} \end{bmatrix}$$

- Making use of the special values for the tangent, leaving out the multiplication by the cosine and combining with angle accumulation, one gets:

$$x^{(i+1)} = x^{(i)} - d_i 2^{-i} y^{(i)}$$
$$y^{(i+1)} = d_i 2^{-i} x^{(i)} + y^{(i)}$$
$$z^{(i+1)} = z^{(i)} - d_i L(i)$$

ROTATION-MODE CORDIC

- Goal is to rotate vector by angle α.
- Initialization:

$$x^{(0)} = x$$
$$y^{(0)} = y$$
$$z^{(0)} = \alpha$$

- Final result:

$$x^{(n)} = K(x\cos(\alpha) - y\sin(\alpha))$$
$$y^{(n)} = K(x\sin(\alpha) - y\cos(\alpha))$$
$$z^{(n)} = 0$$

- Where:

$$K = \prod_{i=1}^{n} \sqrt{1 + 2^{-2i}}$$

- K converges to 1.647.
- Conclusion: the result vector is rotated but scaled version of original vector.

VECTORING-MODE CORDIC

- Determine d_i by an alternative rule: $d_i = -1$ when $y^{(i)} > 0$ and $d_i = +1$ when $y^{(i)} \leq 0$.
- Initialization:

$$x^{(0)} = x$$
$$y^{(0)} = y$$
$$z^{(0)} = 0$$

- Final result:

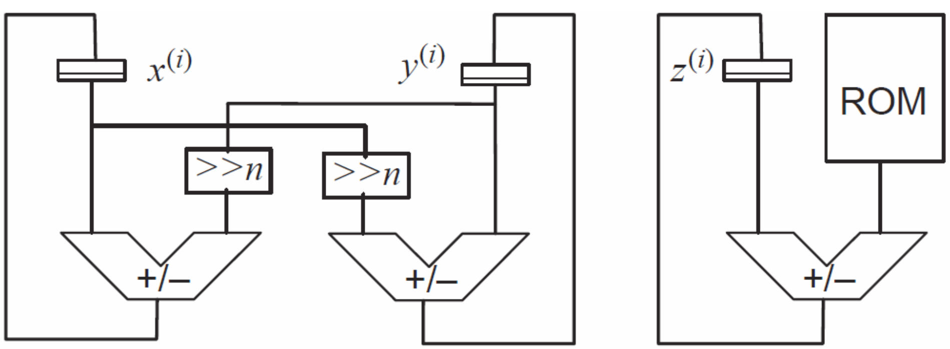
$$x^{(n)} = K\sqrt{x^2 + y^2}$$
$$y^{(n)} = 0$$
$$z^{(n)} = \arctan\left(\frac{y}{x}\right)$$

- This means that the initial vector has been rotated (and scaled) onto the X-axis, while the angle with the X-axis has been computed as well.

BASIC APPLICATIONS OF CORDIC

- *Arctangent*, *vector-magnitude* calculation and *rectangular-to-polar conversion*: direct result of vectoring-mode CORDIC.
- *Polar-to-rectangular conversion*, i.e. from (r, θ) to (x, y) :
  - Set $x^{(0)} = r$, $y^{(0)} = 0$, and $z^{(0)} = \theta$ in rotation mode.
  - Result will be $x = x^{(n)} = Kr\cos(\theta)$, $y = y^{(n)} = Kr\sin(\theta)$.
  - Correction for scaling by K may be necessary (does not require a full-fledged multiplier as K is constant).
- *Sine or cosine* calculation:
  - See above, set $x^{(0)} = 1/K$. Then $x^{(n)} = \cos(\theta)$ and $y^{(n)} = \sin(\theta)$.
- *Beyond the scope of this course:* multiplication, division, hyperbolic functions, etc. (see paper by Andraka).

ARCHITECTURE ITERATIVE CORDIC



$$x^{(i+1)} = x^{(i)} - d_i 2^{-i} y^{(i)}$$
$$y^{(i+1)} = d_i 2^{-i} x^{(i)} + y^{(i)}$$
$$z^{(i+1)} = z^{(i)} - d_i L(i)$$

Controller should take care of initializations, add/subtract decisions, number of iterations, etc.

UNROLLED ARCHITECTURE

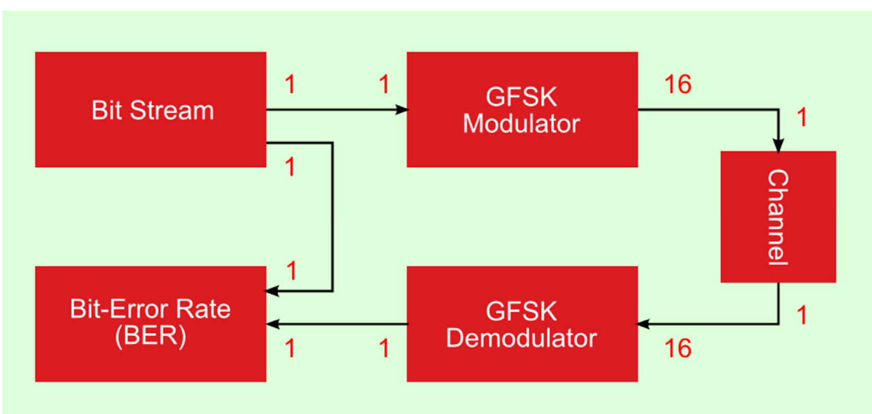
- The iterative architecture requires one clock cycle per iteration.
- It requires a *barrel shifter* to shift operand over a variable number of positions.
- One can also *unroll* the architecture to perform all operations in a single clock cycle:
  - Amounts to instantiate new hardware for each iteration.
  - Possibly adding *pipelining* if the *critical path* becomes too long.
  - The barrel shifter is no longer necessary: each stage in the hardware has a fixed shift which costs just wires.
  - One could also unroll the architecture *partially*.

DESIGN EXAMPLE: GFSK RECEIVER

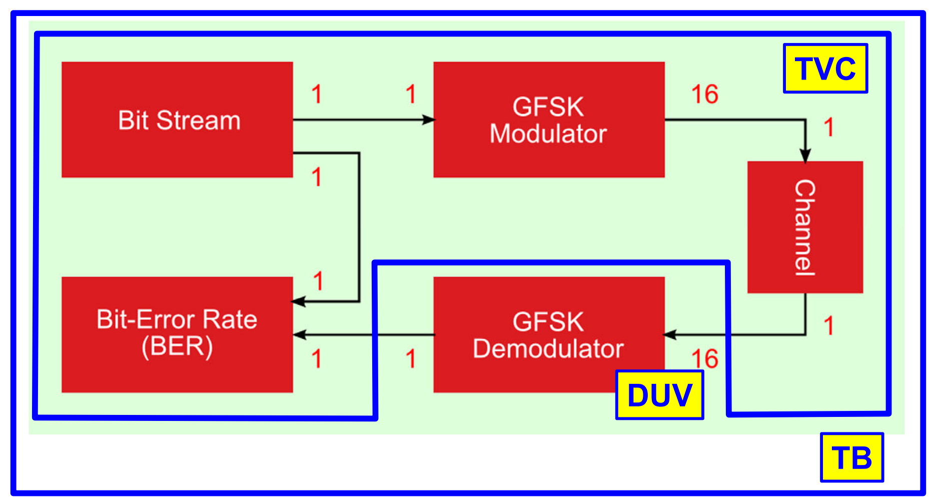
- What is GFSK?
  - *Gaussian frequency shift keying*
  - Method for digital transmission based on frequency modulation (FM).
  - To transmit a 1 carrier frequency is slightly increased and to transmit a 0 the frequency is slightly decreased (or vice versa).
  - The transition steps are smoothed by a Gaussian filter.
  - Found in many standards such as Bluetooth and DECT.
  - Proposed version uses parameters not related to any standard.

GFSK RECEIVER DESIGN APPROACH

- Model entire system: transmitter, receiver, and a channel adding noise (AWGN).
- Leave out analog circuitry for upconversion to RF and downconversion back to IF.
- Use IT++ to set up testbench.
- The testbench computes *bit error rates* (BERs) for different *signal-to-noise ratios* (SNRs).
- Goal is to preserve BER performance when designing hardware.



Modulated signal has 16 samples per transmitted bit.



DUV = design under verification
TVC = test-vector controller; TB = testbench

IMPLEMENTATION ASPECTS

- *Projects focus on designing in Arx.*
- Testbenches for generated C++ and VHDL will be provided.
- As C++ and VHDL behave exactly the same, most simulations will be done in C++ (simulation speed for BER simulations is important).
- C++ testbenches make use of IT++, an open-source library for telecom/signal processing:
  - `http://itpp.sourceforge.net`
  - It provides Matlab-style programming in C++, so vectors, matrices, etc. and lots of powerful functions to manipulate them.

GFSK: MODULATION IN FORMULAE

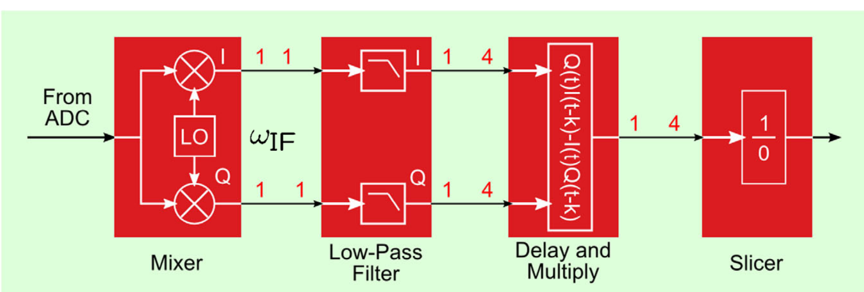
- The modulated signal: $s(t) = A\cos(\omega_{\mathrm{IF}}t + \phi(t))$
- where:
  - A is the constant amplitude
  - ω_{IF} is the *intermediate frequency* (acts as carrier frequency)
  - $\phi(t)$ is the phase deviation, derived from the bit stream
- The phase deviation:

$$\phi(t) = h\pi \int_{-\infty}^{t} \sum_i a_i g(\tau - iT)d\tau$$

- where:
  - h is the modulation index
  - $g(t)$ is a Gaussian-filtered square wave
  - a_i is 1 for a transmitted `1` and -1 for a transmitted `0`.

Study the handout for details!

DEMODULATOR BLOCK DIAGRAM

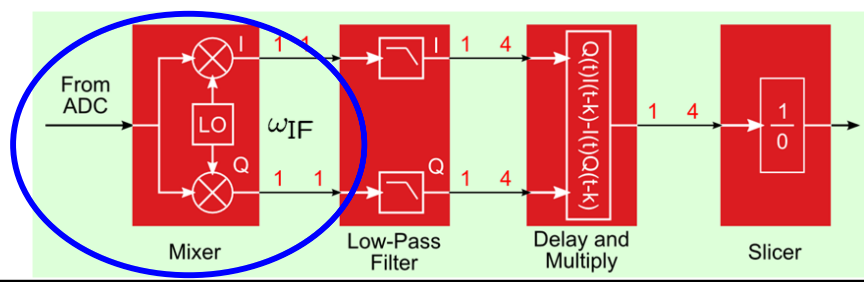


LPF DAM

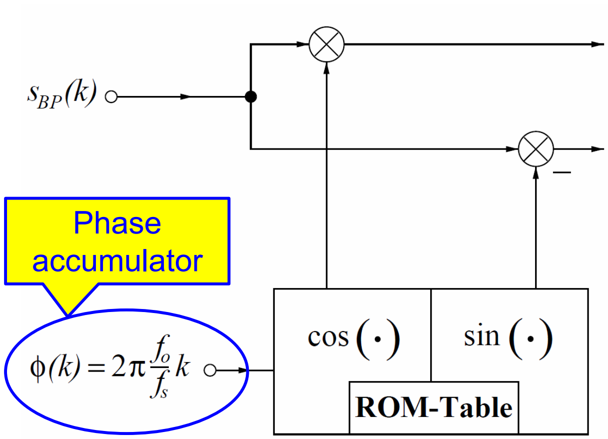
The 16 samples per transmitted bit are first reduced to 4 and later back to 1.

CORDIC FOR DOWNCONVERSION (1)

- *Digital downconversion* is a common operation in digital radio receivers. It is used to shift the carrier frequency of a radio signal (e.g. from IF to baseband) or correct for frequency offset.
- This is done by multiplying an input signal by a sine and cosine of some frequency. Think of the GFSK demodulator.



CORDIC FOR DOWNCONVERSION (2)



$$I(k) \approx s_{BP}(k) \cos\left(2\pi \frac{f_o}{f_s} k\right)$$

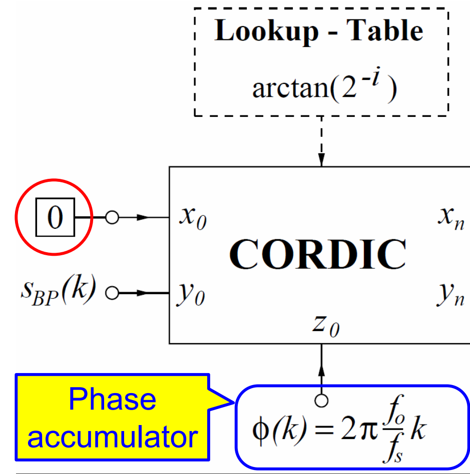
$$Q(k) \approx - s_{BP}(k) \sin\left(2\pi \frac{f_o}{f_s} k\right)$$

Loehning, et al. Figure 1

This solution, requires look-up table (LUT) and 2 multipliers.

CORDIC FOR DOWNCONVERSION (3)

This solution requires just one CORDIC.



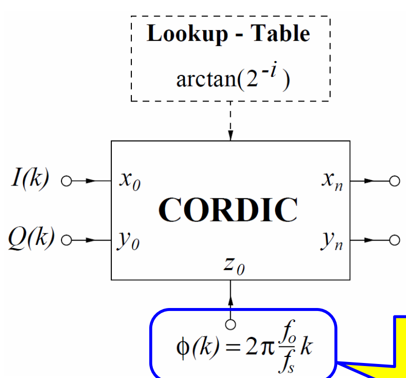
$$Q(k) \approx - A_n s_{BP}(k) \sin\left(2\pi \frac{f_o}{f_s} k\right)$$

$$I(k) \approx A_n s_{BP}(k) \cos\left(2\pi \frac{f_o}{f_s} k\right)$$

Loehning, et al. Figure 2

CORDIC FOR DOWNCONVERSION (4)

IMAGE REJECTION MIXER: "complex" input with in-phase (I) and quadrature (Q) component. CORDIC replaces 4 multipliers!



$$I'(k) \approx A_n\left[I(k) \cos\left(2\pi \frac{f_o}{f_s} k\right) - Q(k) \sin\left(2\pi \frac{f_o}{f_s} k\right)\right]$$

$$Q'(k) \approx A_n\left[Q(k) \cos\left(2\pi \frac{f_o}{f_s} k\right) + I(k) \sin\left(2\pi \frac{f_o}{f_s} k\right)\right]$$

Phase accumulator

Loehning, et al. Figure 4