# DYNAMICS

* Dynamical systems can be described by a *state-space model*, where the state is given by a vector $\boldsymbol{x}$ and the excitation by a vector $\boldsymbol{u}$:

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{F}(\boldsymbol{x}(t), \boldsymbol{u}(t))$$

* A *linear system* is a special case:

$$\frac{d}{dt}\boldsymbol{x}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

* Finding the solution of a linear system is based on finding the eigenvalues of the state transition matrix $A$.





# LINEARIZATION (1)

* Nonlinear systems are difficult to solve in general. Consider the systems without excitation:

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{F}(\boldsymbol{x}(t)),\ \boldsymbol{F}(\boldsymbol{x}(t)) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix}$$

* States $\boldsymbol{x}$ for which $\boldsymbol{F}(\boldsymbol{x}(t)) = 0$ are called *equilibrium points*.
* Suppose that $\boldsymbol{x}_0$ is an equilibrium point. For points $\boldsymbol{x}_0 + \Delta\boldsymbol{x}$ close to this point, using Taylor-series expansion:

$$\frac{d}{dt}(\boldsymbol{x}_0 + \Delta\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}_0) + F'(\boldsymbol{x}_0)\Delta\boldsymbol{x} + \textit{higher order terms}$$

with $F'(x) = \left[\frac{\partial}{\partial x_j} f_i\right]$, the *Jacobian*.





# LINEARIZATION (2)

* We had:

$$\frac{d}{dt}(\boldsymbol{x}_0 + \Delta\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}_0) + F'(\boldsymbol{x}_0)\Delta\boldsymbol{x} + \textit{higher order terms}$$

* Neglecting the higher order terms:

$$\frac{d}{dt}\Delta\boldsymbol{x} = F'(\boldsymbol{x}_0)\Delta\boldsymbol{x}$$

which is a linearized system. The eigenvalues of the Jacobian around equilibrium points describes the local behavior.





# STABILITY

* Consider an initial state $\boldsymbol{x}(0)$ in the state space close to an equilibrium point $\boldsymbol{x}_0$: $\|\boldsymbol{x}(0) - \boldsymbol{x}_0\| < \delta$.
  + The system is *uniformly* stable if there exists a $\delta$ for any given $\epsilon$ such that $\|\boldsymbol{x}(t) - \boldsymbol{x}_0\| < \epsilon$ for all $t$.
  + The system is *asymptotically* stable if a $\delta$ can be found such that $\lim_{t \to \infty} \boldsymbol{x}(t) = \boldsymbol{x}_0$.
  + The system is *marginally* stable if the system is uniformly stable, but not asymptotically stable.
  + The system is *unstable* if it is not uniformly stable.

# LYAPUNOV STABILITY (1)

* Given the system for which $\frac{d}{dt}\mathbf{x}(t) = \mathbf{F}(\mathbf{x}(t))$, suppose that a function $V(\mathbf{x})$ exists with the following properties in a neighborhood around equilibrium points $\mathbf{x}_0$:
  + $V(\mathbf{x})$ is continuous and has partial derivatives with respect of all elements of $\mathbf{x}$.
  + $V(\mathbf{x}_0) = 0$ for equilibrium points $\mathbf{x}_0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}_0$.
* The system is uniformly stable if: $\frac{d}{dt}V(\mathbf{x}) \leq 0$ and asymptotically stable if $\frac{d}{dt}V(\mathbf{x}) < 0$.





# LYAPUNOV STABILITY (2)

* $\frac{d}{dt}V(\mathbf{x}) < 0$, means:

$$\frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} = \nabla V \cdot \frac{d}{dt}\mathbf{x} < 0$$

* Because the gradient is always perpendicular to the contour, this means that $\mathbf{x}$ moves closer to the equilibrium point.